NEWTON-RAPHSON METHOD

Introduction
Equation of the type \( f(x) = 0 \) can be solved graphically by plotting the curve \( y = f(x) \) and find its \( x \)-intercepts.
The accuracy of the solutions is limited by the scale of the graph.

A method which enables the roots to be calculated to any desired accuracy is the **Newton-Raphson method** or simply **Newton’s method**.
If $x_1$ is an approximate value for a real root $x = a$ of the equations $f(x) = 0$, then a closer approximation $x_2$ may be obtained for this root using the formula.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
The formula can be used repeatedly to find successively closer approximations, $x_3, x_4$, etc., to the root $x = a$.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

The formula can be applied repeatedly until the desired accuracy has been met.
In general the Newton’s method can be summarized as the following equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton’s formula can generally be used for any polynomial or non-polynomial function.

First approximations can be obtained by drawing the graph or examining the function.

In general, the closer the first approximations is to the real root the sooner the terms $x_1, x_2, x_3$, etc will converge to the root.
Example 1
Show that the equation $x^3 - 3x - 3 = 0$ has a root between $x = 2$ and $x = 3$. Use Newton’s method to determine the root correct to 4 significant figures.

$$f(x) = x^3 - 3x - 3$$

$$f(2) = 2^3 - 3(2) - 3 =$$

$$f(3) = 3^3 - 3(3) - 3 =$$

It can be seen that $f(2)$ and $f(3)$ have opposite changes signs. In other words, $f(x) = 0$ for a certain value between $x = 2$ and 3 and a root exists between $x = 2$ and $x = 3$. 
\[ f(x) = x^3 - 3x - 3 \]

\[ f'(x) = 3x^2 - 3 \]

\[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^3 - 3x_1 - 3}{3x_1^2 - 3} \]

Choose \( x_1 = 2.5 \), then

\[ x_2 = \]

\[ x_3 = \]

\[ x_4 = \]
Example 2
Show that the equation $x^5 - 2x - 3 = 0$ has a root between $x = 1.4$ and $x = 1.6$. Use Newton’s method to determine the root correct to 4 significant figures.

$$f(x) = x^5 - 2x - 3$$
$$f(1.4) = 1.4^5 - 2(1.4) - 3 =$$
$$f(1.6) = 1.6^5 - 2(1.6) - 3 =$$

It can be seen that $f(1.4)$ and $f(1.6)$ have opposite changes signs. In other words, $f(x) = 0$ for a certain value between $x = 1.4$ and $1.6$ and a root exists between $x = 1.4$ and $x = 1.6$. 

\[ f(x) = x^5 - 2x - 3 \]
\[ f'(x) = 5x^4 - 2 \]
\[
\begin{align*}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^5 - 2x_1 - 3}{5x_1^4 - 2}
\end{align*}
\]

Choose \( x_1 = 1.5 \), then

\( x_2 = \) 
\[ x_3 = \]
\[ x_4 = \]
Example 3

Use Newton’s method to determine the root of the equation \( \cos x = 1.5 \tan x \) correct to 4 decimal places using \( x_1 = \pi/4 \).

(Note: radian should be used for the argument of the trigonometric function but not in degree)
Evaluation of $N^k$

The value of $N^k$ can be found using the Newton’s method by rearranging the number in the form $f(x) = 0$.

Let

$x = N^k$

Then

\[ x^k = N \]

\[ x^k - N = 0 \]

Finally solve $f(x) = 0$ by Newton’s method for

\[ f(x) = x^k - N \]
Example 4
Use Newton’s method to evaluate $\sqrt[3]{19}$ correct to 5 decimal places.
Example 5

Use Newton’s method to evaluate \( \frac{1}{2.8} \) correct to 5 decimal places.
NUMERICAL INTEGRATION

Introduction

The area under a curve between limits $x = a$ and $x = b$ can be given by the integral

$$\text{Area} = \int_a^b y \, dx$$
In some cases, it is not possible to evaluate the integral by direct mathematical integration.

Under such circumstances, we can estimate the numerical value of the integral by

1. Dividing the required area in a number of strips.

2. Find the area of each strip.

3. Add the areas up.
There are several methods for doing this numerical integration. In our course, we are going to discuss the following two

**Trapezium (or Trapezoidal) Rule**

**The Simpson’s Rule**
Trapezoidal Rule

In this method, the area is divided into $n$ vertical strips of equal width $w$.

$$w = \frac{b - a}{n}$$

$x_0 = a \ x_1 \ x_2 \ x_3 \ x_{n-1} \ x_n = b$

$$w \ w \ w \ w$$
Each vertical strip is treated as a trapezium to find the approximate area.

For the third strip

$$A = \frac{w}{2}(y_2 + y_3)$$
The integral can be approximated by the sum of areas of the strips.

\[
\int_{a}^{b} y \, dx = \frac{w}{2} (y_{o} + y_{1}) + \frac{w}{2} (y_{1} + y_{2}) + \frac{w}{2} (y_{2} + y_{3}) \\
+ \ldots + \frac{w}{2} (y_{n-1} + y_{n})
\]
Hence

\[ \int_{a}^{b} y \, dx = \frac{w}{2} \left( y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n \right) \]

The trapezoidal rule is hence

\[ \int_{a}^{b} y \, dx = \frac{b - a}{2n} \left( y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n \right) \]
Example 6

Find \( \int_0^3 e^x \, dx \) using trapezoidal rule by dividing the area into 6 strips. Give your answer correct to 3 decimal places.

<table>
<thead>
<tr>
<th>number of strips =</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = )</td>
</tr>
<tr>
<td>( w = )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
<th>( m )</th>
<th>( my_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>2</td>
<td></td>
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<td>2</td>
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<td>2</td>
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<tr>
<td>3</td>
<td></td>
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<td>2</td>
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<td>2</td>
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</tr>
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<td>5</td>
<td></td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td></td>
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</tr>
</tbody>
</table>


Example 7

Find \( \int_{2}^{3} \ln x \, dx \) by dividing the area into 8 strips using the Trapezoidal rule.

Give your answer correct to 4 decimal places.
Example 8

Find \( \int_{1}^{5} \frac{x}{x+3} \, dx \) by dividing the area into 8 strips using Trapezoidal rule.

Give your answer correct to 4 decimal places.
Example 9

Find \( \int_{1}^{2} \frac{\ln x}{\sqrt{9 - x^2}} \, dx \) by dividing the area into 8 strips using Trapezoidal rule.

Give your answer correct to 4 decimal places.
Example 10

Find \[ \int_{0}^{4} x^2 e^{-x} \, dx \] using Trapezoidal rule by dividing the area into

(a) 8 strips

(b) 10 strips

Give your answer correct to 4 decimal places.
Simpson’s Rule

There is another method, called Simpson’s Rule, that can be applied to find the numerical value of a definite integral by the area

\[ \text{Area} = \int_a^b y \, dx \]
In this method, the area is divided into $n$ vertical strips of equal width $w$.

*Note that*: $n$ must be an even number.

$$w = \frac{b - a}{n}$$
Each two vertical strips are combined and treated under a parabola to find the approximate area.

\[ A = \frac{w}{3}(y_2 + 4y_3 + y_4) \]
The integral can be approximated by the sum of areas of the strips.

\[ \int_{a}^{b} y \, dx = \frac{w}{3} (y_o + 4y_1 + y_2) + \frac{w}{3} (y_2 + 4y_3 + y_4) \]

\[ + \ldots + \frac{w}{3} (y_{n-2} + 4y_{n-1} + y_n) \]
Hence

\[ \int_a^b y \, dx = \frac{w}{3} \left( y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_{n-1} + y_n \right) \]

The Simpson’s rule is hence

\[ \int_a^b y \, dx = \frac{b-a}{3n} \left( y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_{n-1} + y_n \right) \]
Example 11
Find \( \int_{0}^{3} e^x \, dx \) using Simpson’s rule by dividing the area into 6 strips. Give your answer correct to 3 decimal places.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>( x_n )</td>
<td>( y_n )</td>
</tr>
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<td>1</td>
</tr>
<tr>
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<td></td>
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<td>2</td>
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<td>4</td>
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<tr>
<td>4</td>
<td>2</td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

\[ \text{total} \]

number of strips = 6
Example 12

Find $\int_{2}^{3} \ln x \, dx$ by dividing the area into 8 strips using Simpson’s rule.

Give your answer correct to 4 decimal places.
Example 13

Find \( \int_{2}^{4} x \ln x \, dx \) by dividing the area into 8 strips using (a) Trapezoidal rule;

(b) Simpson’s rule.

Give your answer correct to 4 decimal places.
Comparison between Trapezoidal rule and Simpson’s Rule

In general, the result obtained by using the Simpson’s rule is more accurate than that obtained by using the Trapezoidal rule.

Hence the Simpson’s rule is preferable if accuracy is concerned.
Example 14

Find \( \int_{0}^{2} e^{x} \, dx \) using Simpson’s rule by dividing the area into

(a) 6 strips 
(b) 8 strips 
(c) 10 strips.

Give your answer correct to 4 decimal places.
Accuracy of Answers Using Simpson’s Rule

It can be seen that the result obtained by using the Simpson’s rule is more accurate if more strips are used.

A simple criteria for determining the accuracy of the result is:

If an increase in the number of strips does not involve a change in the answer to a degree of accuracy, then we may rely on the result to that degree of accuracy.
For example, for the integral
\[ \int_0^2 e^x \, dx \]
the following results are obtained when Simpson’s rule is used in previous example:

<table>
<thead>
<tr>
<th>Number of strips</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6.389</td>
</tr>
<tr>
<td>8</td>
<td>6.389</td>
</tr>
<tr>
<td>10</td>
<td>6.389</td>
</tr>
</tbody>
</table>

We can say that the result is 6.389 correct to 3 decimal places or 4 significant figures.